Name: NetId:

FINAL 10AM-11.59AM (2H00MIN)
ONE HANDWRITTEN TWO-SIDED CHEATSHEET ALLOWED

Contents

Problem 0: Let $\mathcal{X}$ be a finite state space. Let f, Z_0 be a random mapping representation for an irreducible transition matrix $P(\cdot,\cdot)$, in the sense that $f : \mathcal{X} \times [0,1] \to \mathcal{X}$ is deterministic, Z_0 is a random variable, and the random function $f_0 : x \mapsto f(x, Z_0)$ satisfies $\mathbf{P}(f_0(x) = y) = P(x,y)$ for all $x, y \in \mathcal{X}$.

Let Y be independent of (Z_0, f_0). Show that if $f_0(Y)$ and Y have the same distribution, then the distribution of Y is stationary for P.

Problem 1: Provide a construction of a 1-dimensional Poisson Point Process on $[0,1]$ with intensity $\lambda(x) = x^2$ on $[0,1]$. You must clearly explain the construction, but you need not prove that the resulting point process is a Poisson Point Process with the desired intensity.

Problem 2(a): Let $(T_k)_{k\geq 1}$ be a sequence of iid Exp(1) random variable, and let $(B_k)_{k\geq 1}$ be a sequence of Bernoulli(p) random variables. Derive the distribution of T_τ where $\tau = \min\{t \geq 1 : B_t = 1\}$.

Problem 2(b): Let T_τ be the same random variable as in the previous question, and let $S_1, ..., S_n$ be iid random variables with the same distribution as T_τ. What is the distribution of $\min_{i=1,\dots,n} S_i$?

Problem 2(c): You volunteer at a marathon to distribute water bottles to runners; three volunteers (including you) distribute bottles to runners arriving at this table. Now the marathon starts: n runners start running towards your table, each of them reaching your table in Exp(1) time independently of other runners. Upon arrival one of the three volunteers (uniformly at random) gives the runner a water bottle. What is the distribution of the time at which the first runner picks up a bottle from you?

Problem 3(a): Draw a graph for a few vertices such that the simple random walk on this graph is an irreducible and aperiodic chain. Give a short explanation as to why the chain is irreducible and aperiodic.

Problem 3(b): Prove that a simple random walk on a finite connected graph cannot be 3-periodic.

Problem 3(c): Provide an example of a matrix P of transition probabilities on $\mathcal{X} = \{1, 2, 3, 4\}$ such that the chain is NOT irreducible, and such that the chain has at least 2 different stationary distributions.

Problem 3(d): Provide an example of a matrix P of transition probabilities on $\mathcal{X} = \{1, 2, 3, 4\}$ such that the chain is NOT irreducible, but the chain still has a unique stationary distribution.

Problem 4: setup. Consider a connected undirected graph $G = (V, E)$. We consider a base chain on $\mathcal{X} = V \times V$ that consists of two particles moving as a simple random walk on the graph, independently of each other. In other words

$$\Psi\Big((u_1, u_2), (v_1, v_2)\Big) = 1/\big(|N(u_1)|\,|N(u_2)|\big) \text{ if both } v_1 \in N(u_1), v_2 \in N(u_2),$$

and

$$\Psi\Big((u_1, u_2), (v_1, v_2)\Big) = 0 \text{ if } v_1 \notin N(u_1) \text{ or } v_2 \notin N(u_2),$$

for any four vertices u_1, u_2, v_1, v_2, where $N(v) \subset V$ is the set of neighbors of a vertex v and $|N(v)|$ the cardinality of $N(v)$. (A vertex v is not a neighbor of itself so that $v \notin N(v)$ always holds.) We are interested in constructing a Markov Chain on $\mathcal{X} = V \times V$ with stationary distribution

$$\forall (u_1, u_2) \in V \times V, \qquad \pi((u_1, u_2)) = \begin{cases} 0 & \text{if } u_1 = u_2, \\ 1/Z & \text{if } u_1 \neq u_2, \end{cases}$$

where $Z > 0$ is a normalizing constant not depending on u_1, u_2.

Problem 4(a): Is the base chain symmetric? (Prove or disprove).

Problem 4(b): If you apply the Metropolis scheme to construct a Chain with stationary distribution π from the base chain Ψ, what is the acceptance probability

$$a\Big((u_1, u_2), (v_1, v_2)\Big) \tag{1}$$

to accept a move from the base chain? Clearly write and simplify $a((u_1, u_2), (v_1, v_2))$ and verify that the detailed balance equations are satisfied.