HOMEWORK 9

654 STOCHASTIC PROCESSES

Note for the current and future homework: due to a large class size and time constraints, not all exercises will be graded.

As you are the first class to go through these homework assignments, they may contain typos/ambiguities. Feel welcome to contact us if you see a typo or have doubt about other issues.

Exercises from Durett's book. See https://services.math.duke.edu/~rtd/EOSP/EOSP2E.pdf

- 2.46
- 2.49
- 2.52
- 2.59
- 2.60
- 2.55 (Optional)
- 2.56 (Optional)
- 2.61 (Optional)

Construction of non-homogeneous Poisson process by change of variable. Let $\lambda : [0, +\infty) \to (0, +\infty)$ be a continuous function and let $\{N(t), t \geq 0\}$ be a Poisson process with rate 1 and let $(T_1, T_2, ...)$ be the jump times of $N(\cdot)$.

1. Assume that there exists a continuously differentiable and strictly increasing function $\Lambda : [0, +\infty) \to [0, +\infty)$ so that $(\Lambda^{-1}(T_1), \Lambda^{-1}(T_2), \Lambda^{-1}(T_3), ...)$ are the jump times of a non-honogeneous Poisson process with rate $\lambda(\cdot)$. Find the relationship between λ and Λ and show that Λ is unique.
2. Show that if $\Lambda(\cdot)$ is the function found in the previous question, then $(\Lambda^{-1}(T_1), \Lambda^{-1}(T_2), \Lambda^{-1}(T_3), ...)$ are the jump times of a non-honogeneous Poisson process with rate $\lambda(\cdot)$.

Discrete Markov Chains from Poisson processes. Consider phone calls that arrive at times $S_1, S_2, S_3, ...$ as a one-dimensional Poisson process with rate λ on $[0, +\infty)$, this process is $N(\cdot)$ and we write $N(t) = N([0, t])$ for simplicity. The i-th call lasts $Y_i \sim Exp(\mu)$ units of time, independently of the call arrivals and independently of the other call lengths.

Let $\{Q(t), t \geq 0\}$ be the number of calls still happening at time t, and let $T_1, T_2, T_3...$ be the points of discontinuity of Q (i.e., the time points when a call starts or a call ends). We construct Q so that it is right continuous: Q is continuous on $[T_k, T_k + \epsilon]$ for small enough ϵ.

1. *(Memoryless property)* For a fixed t, what is the joint distribution of

$$S_{n+1} - t, \max(0, S_1 + Y_1 - t), \max(0, S_2 + Y_2 - t), ..., \max(0, S_n + Y_n - t)$$

conditionally on the event

$$N(t) = n, Q(t) = q, I_{S_1+Y_1 \leq t} = \delta_1, I_{S_2+Y_2 \leq t} = \delta_2, ...I_{S_n+Y_n \leq t} = \delta_n$$

where $n, q \geq 0$ are deterministic integers and $\delta_1, ..., \delta_n$ are in $\{0, 1\}$ such that $\sum_{i=1}^{n} \delta_i = q$ (the δ's indicate which calls are still ongoing at time t).

2. Define a discrete-time process $(X_k)_{k=1,2,3,...}$ by $X_k = Q(T_k)$, the number of calls still happening at the k-th point of discontinuity T_k. Show that $(X_k)_{k\geq 1}$ is a Markov Chain on the non-negative integers, and find its matrix of transition probabilities $P(\cdot, \cdot)$.
3. Find π the stationary distribution of $P(\cdot, \cdot)$.
4. Compare π with the distribution of long-run limit ($t \to +\infty$) of the number of calls still happening at time t, computed in the lecture.

Programming.

1. Create a program that generates a Poisson Point process $N(\cdot)$ in the square $[0, 1]^2$, with intensity function $\lambda(x, y) = 180yx(1 - x)^2$. You may use python/R functions to generate iid Beta distributions.
2. Can you provide a different method to generate the same process?
3. What is the joint distribution of the number of points of the four sub-squares

$$N([0, 0.5] \times [0, 0.5]), N([0.5, 1] \times [0, 0.5]), N([0, 0.5] \times [0.5, 1]), N([0.5, 1] \times [0.5, 1])?$$

By sampling many iid copies of the Poisson process, plot histograms that approximate the pmf of these four random variables, and compare with the real pmf.