HOMEWORK 3

654 STOCHASTIC PROCESSES

Note for the current and future homework: due to a large class size and time constraints, not all exercises will be graded.

As you are the first class to go through these homework assignments, they may contain typos/ambiguities. Feel welcome to contact us if you see a typo or have doubt about other issues.

Exercises from the Book page 18.

- 1.10 (Optional)
- 1.12 (Optional)
- 1.13
- 1.14

The Strong Markov property. Let $\mathcal{X}$ be finite and consider Markov chain $\{X_t, t = 0, 1, 2, ...\}$ on $\mathcal{X}$ with transition matrix P.

Let τ be a stopping time, that is, a random variable valued in $\{0, 1, 2, ...\} \cup \{+\infty\}$ such that for any $t \geq 0$, the event $\{\tau = t\}$ is determined by $(X_0, ..., X_t)$. Equivalently, τ is a stopping time if for any $t \geq 0$, there exists a function $p_t : \mathcal{X}^{t+1} \to \{0, 1\}$ such that the event $\{\tau = t\}$ occurs if and only if $p_t(X_0, ..., X_t) = 1$. (You may use these functions $p_t()$ for the following questions).

Let $x \in \mathcal{X}$; we assume here that $\mathbf{P}[X_0 = x] = 1$.

1. Let $I_{\{\tau<+\infty\}}$ be the indicator function of the event $\{\tau < +\infty\}$. Express this indicator function as an infinite sum of indicator functions of events $\{\tau = t\}$.
2. If $f : \mathcal{X}^{m+1} \to \{0, 1\}$ is a function, define the function $F : \mathcal{X} \to \mathbb{R}$ by $F(z) = \mathbf{E}_z[f(X_0, ..., X_m)]$ for any $z \in \mathcal{X}$. Show that
$$\mathbf{E}_x[I_{\{\tau<+\infty\}} f(X_\tau, X_{\tau+1}, ..., X_{\tau+m})] = \mathbf{E}_x[I_{\{\tau<+\infty\}} F(X_\tau)].$$
You may invoke Proposition A.11 of the book to argue about changing the order of summations (or limits) and expectations.
3. Simplify the right hand side of the previous display in the case where $\mathbf{P}_x[X_\tau = z] = 1$ (for instance, τ could be the first hitting time of z).
4. Show that if now $f : \mathcal{X}^{m+1} \to \mathbb{R}$, then f can be written as a linear combination of a finite number of functions $\mathcal{X}^{m+1} \to \{0, 1\}$. Generalize question 2 to any $f : \mathcal{X}^{m+1} \to \mathbb{R}$.
5. Let $C_m = \mathcal{X}^m$ and let g be a function $\cup_{m=0}^{+\infty} C_m \to \mathbb{R}$. In words, g is a function that takes as input arbitrarily but finitely many elements of $\mathcal{X}$, and

the notation $g(x_0, x_1, ..., x_t)$ makes sense for any $t \geq 0$ and any $x_0, ..., x_t \in \mathcal{X}$. Show that

$$\mathbf{E}_x[g(X_0, ..., X_\tau)I_{\{\tau<+\infty\}}f(X_\tau, X_{\tau+1}, ..., X_{\tau+m})] = \mathbf{E}_x[g(X_0, ..., X_\tau)I_{\{\tau<+\infty\}}F(X_\tau)]$$

and simplify the statement in the case $\mathbf{P}_x[\tau < +\infty, X_\tau = z] = 1$.

6. Assume that $\mathbf{P}_x[\tau < +\infty] = 1$ and define the sequence of random variables $Y_t = X_{t+\tau}$ for all $t \geq 0$. Show that Y_t is a Markov chain with transition matrix P and initial distribution $\mathbf{P}[Y_0 = y] = \mathbf{P}_x[X_\tau = y]$ for any $y \in \mathcal{X}$.
7. Let $\tau_x^0, \tau_x^1, ..., \tau_x^r, ...$ be the return times to x, i.e., $\tau_x^0 = 0$ and $\tau_x^r = \inf\{t > \tau_x^{r-1} : X_t = x\}$. Using the previous question, explain why $\mathbf{E}_x[\tau_x^r] = r\mathbf{E}_x[\tau_x^1]$ for any $r > 0$. Use Lemma 1.13 to conclude that $\mathbf{P}_x(\tau_x^r < +\infty) = 1$. Show that for any $f, g \in \cup_{m=0}^{+\infty} C_m \to \{0, 1\}$,

$$\begin{aligned}\mathbf{E}_x[g(X_0, ..., X_{\tau_x^1})f(X_{\tau_x^1}, ..., X_{\tau_x^r})] &= \mathbf{E}_x[g(X_0, ..., X_{\tau_x^1})]\mathbf{E}_x[f(X_{\tau_x^1}, ..., X_{\tau_x^r})], \\ &= \mathbf{E}_x[g(X_0, ..., X_{\tau_x^1})]\mathbf{E}_x[f(X_0, ..., X_{\tau_x^{r-1}})].\end{aligned}$$

Conclude that the events $\{g(X_0, ..., X_{\tau_x^1}) = 1\}$ and $\{f(X_{\tau_x^1}, ..., X_{\tau_x^2}) = 1\}$ are independent under $\mathbf{P}_x$.
8. With the notation of the previous question, conclude that $h(X_0, ..., X_{\tau_x^1})$ and $h(X_{\tau_x^1}, ..., X_{\tau_x^2})$ are iid for any function $h : \cup_{m=1}^{+\infty} C_m \to \mathbb{R}$.

The reader with some measure-theoretic background is invited to look at Proposition A.19 in the book.

On Proposition 1.14(i).

1. Generalize Proposition 1.14(i) in the book to the case τ_z^+ is replaced by a stopping time τ such that $\mathbb{P}_z(\tau < +\infty, X_\tau = z) = 1$.
   a. State the generalization of Proposition 1.14(i) for such τ.
   b. Does the proof of the book need to be modified if τ is as above?
2. Assume that the chain satisfies $X_0 = z$ with probability 1 and that the chain is irreducible. Let τ be the first time that the chain returns to z after having visited every other states at least once, i.e.,

$$\tau = \inf\left\{t > 0 : X_t = z \text{ and } \mathcal{X} \subseteq \{X_1, ...X_t\}\right\}.$$

Is τ a stopping time and does it satisfy $\mathbb{P}_z(\tau < +\infty, X_\tau = z) = 1$?

The ergodic Theorem (Optional). Let $\{X_t, t = 0, 1, 2, 3, ...\}$ be a Markov chain on a finite set $\mathcal{X}$. Assume that the chain is irreducible with stationary distribution π.

Let $M > 0$. For any bounded function $f : \mathcal{X} \to [-M, M]$, define

$$E_\pi(f) = \sum_{x\in\mathcal{X}} \pi(x)f(x).$$

Let $\tau_x^0, \tau_x^1, ..., \tau_x^r, ...$ be the return times to x, i.e., $\tau_x^0 = 0$ and $\tau_x^r = \inf\{t > \tau_x^{r-1} : X_t = x\}$.

Recall the Strong Law of Large Numbers, which states that if $(Z_n)_{n=1,2,...}$ are mutually independent and identically distributed with $\mathbf{E}[|Z_1|] < +\infty$ then

$$\mathbf{P}\left(\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} Z_i = \mathbf{E}[Z_1]\right) = 1.$$

By definition, *Mutual independence* requires that for any finite subset $\{i_1, ..., i_N\}$ of non-negative integers, one has for any real numbers $a_1, ..., a_n$

$$\mathbf{P}\left[\cap_{j=1}^{N}\{Z_j \leq a_j\}\right] = \prod_{j=1}^{N} \mathbf{P}[Z_j \leq a_j].$$

(The Strong Law of Large Numbers also holds if the sequence of variables is only pairwise independent, which is a weaker requirement that mutually independent).

1. Show that if two events A, B have probability 1, then $A \cap B$ also has probability 1.
2. For any $k \geq 0$, let $S_k = \sum_{t=\tau_x^k}^{\tau_x^{k+1}-1} f(X_t)$. Using the results from the "Strong Markov Property" exercise, explain why the random variables $(S_k)_{k=0,1,2,3...}$ are mutually independent and identically distributed.
3. Show that
$$\mathbf{P}_x\left(\lim_{n\to+\infty} \frac{\sum_{t=0}^{\tau_x^n-1} f(X_t)}{\tau_x^n} = L\right) = 1 \qquad \text{where} \qquad L = \frac{\mathbf{E}_x\left[S_0\right]}{\mathbf{E}_x[\tau_x^1]}$$
4. Using Proposition 1.14 from the book, show that L from the previous question is equal to $E_\pi(f)$.
5. Let $(a_k)_{k=0,1,2,...}$ be a bounded sequence in $[-M, M]$ and let $(T_k)_{k=0,1,2,3...}$ be a sequence of integers such that $\lim_{k\to\infty} T_k/T_{k+1} = 1$ and $\lim_{k\to\infty} T_k = +\infty$. Show that for any real number L,
$$\lim_{k\to+\infty} \frac{a_1 + ... + a_{T_k}}{T_k} = L \qquad \text{implies} \qquad \lim_{T\to+\infty} \frac{a_1 + ... + a_T}{T} = L.$$
6. Why does $\tau_x^n \geq n$ hold? Conclude that
$$\mathbf{P}_x\left(\lim_{T\to+\infty} \frac{\sum_{t=0}^{T-1} f(X_t)}{T} = E_\pi(f)\right) = 1.$$
7. Generalize the previous result to the case where X_0 has distribution μ.
8. Let $x_0 \in \mathcal{X}$. State the result in the special case of the function f such that $f(y) = 1$ if $y = x_0$ and $f(y) = 0$ for $y \neq x_0$. On average, which proportion of time does the chain spend at x_0?